

# KOSZUL THEOREM FOR $\mathcal{S}$ -LIE COALGEBRAS

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ABSTRACT. For a symmetry braid  $\mathcal{S}$ -Lie coalgebras, as a dual object to algebras introduced by Gurevich [2], are considered. For an Young antisymmetrizer  $Y(-\mathcal{S})$  an  $\mathcal{S}$ -exterior algebra is introduced. From this differential point of view  $\mathcal{S}$ -Lie coalgebras are investigated. The dual Koszul theorem in this case is proved.

## 1. INTRODUCTION

For a symmetric braid  $\mathcal{S}$  we wish to investigate  $\mathcal{S}$ -Lie coalgebras as a dual object to  $\mathcal{S}$ -Lie algebras introduced by Gurevich [2]. With an  $\mathcal{S}$ -Lie coalgebra we consider an  $\mathcal{S}$ -exterior algebra over a coalgebra given by an Young antisymmetrizer  $Y(-\mathcal{S})$  [1], [5]. The aim of this paper is to find out a kind of the Koszul theorem [3] for  $\mathcal{S}$ -Lie coalgebras. This means that the Jacobi condition for an  $\mathcal{S}$ -Lie coalgebra is equivalent to the differential condition  $\hat{d}^2 = 0$  for a derivation of an  $\mathcal{S}$ -exterior algebra. We are interested in finding the cohomological meaning of the Jacobi condition for  $\mathcal{S}$ -Lie (co)algebras [2].

In section 1 we review definitions in a braided monoidal category  $X$ . In the paper we use the following notations. Let  $k$  be a field and  $\Gamma$  be a  $k$ -space, a dual  $k$ -space  $\Gamma^* = \text{Hom}(\Gamma, k)$  and  $T\Gamma$  means a tensor algebra for  $k$ -space  $\Gamma$ . We consider a strict monoidal category  $X$  generated by one object  $\Gamma \in \text{Obj}(X)$ , and  $\text{Mor}(X)$  is a set of morphisms. By  $d \in \text{der}(T\Gamma)$  let us denote a set of derivations of a free graded algebra  $T\Gamma$ .

For the symmetry braid  $\mathcal{S}$  we have a symmetric monoidal category and the definition of a Lie (co)algebra is given over this category. For the braid group we consider the notion of an Young antisymmetrizer. As an example we have an  $\mathcal{S}$ -antisymmetrizer  $Y(-\mathcal{S})$ . An  $\mathcal{S}$ -exterior algebra is a factor algebra of a free algebra  $T\Gamma$  and an ideal is equal to a kernel of the operator  $Y(-\mathcal{S})$ .

Section 2 contains a brief summary of definitions and remarks of a Lie (co)algebra over a monoidal category. We will touch only a few aspects of the theory of Lie coalgebras. for a more complete theory see [6]. New is the dual Koszul theorem for

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Lie coalgebras. This is possible, because we show that for any derivation of a free tensor algebra exists a factor derivation of an exterior algebra.

In section 3 we consider the notion of  $\mathcal{S}$ -Lie coalgebras over an  $\mathcal{S}$ -braided monoidal category. The basic idea is that the morphism condition for a comultiplication guarantees the existence of a factor derivation of a free tensor algebra factor by the kernel of the Young antisymmetrizer  $Y(-\mathcal{S})$ .

The main result of this paper is the dual Koszul theorem for  $\mathcal{S}$ -Lie coalgebras as the possible form of the Jacobi identity [7]. Let  $\Lambda_{\mathcal{S}}\Gamma = \Gamma^{\otimes}/\ker Y(-\mathcal{S})$  and  $\hat{d} \in \text{der}(\text{TT} / \ker Y(-\mathcal{S}))$ ,  $\mathcal{S}_1 = \mathcal{S} \otimes \text{id}$  and  $\mathcal{S}_2 = \text{id} \otimes \mathcal{S}$ . For an  $\mathcal{S}$ -exterior algebra  $\mathcal{E}_{\mathcal{S}}(\Gamma) = (\Lambda_{\mathcal{S}}\Gamma, \hat{d}_{\Delta})$  over a  $\mathcal{S}$ -Lie coalgebra  $\text{SLC} = (\Gamma, \Delta, \mathcal{S})$  holds the equation  $\hat{d}_{\Delta}|_{\Gamma} = \Delta$ .

**Theorem.(Koszul Theorem for  $\mathcal{S}$ -Lie coalgebras).**

For the Young antisymmetrizer  $Y(-\mathcal{S})$  let  $\mathcal{E}_{\mathcal{S}}(\Gamma)$  be the  $\mathcal{S}$ -exterior algebra over the  $\mathcal{S}$ -Lie coalgebra  $\text{SLC}$ . Then the following conditions are equivalent

- $\hat{d}^2 = 0 \in \Gamma^{\wedge 3}$ ,
- $\text{im}(\hat{d}_{\Delta})^2 \subset \ker Y_3(-\mathcal{S})$ ,
- $(\text{id} + \mathcal{S}_1\mathcal{S}_2 + \mathcal{S}_2\mathcal{S}_1) \circ (\Delta \otimes \text{id}) \circ \Delta = 0$ ,
- $(\Delta \otimes \text{id}) \circ \Delta - (\text{id} \otimes \Delta) \circ \Delta - (\text{id} \otimes \mathcal{S}) \circ (\Delta \otimes \text{id}) \circ \Delta = 0$ .

## 2. NOTATIONS IN BRAIDED MONOIDAL CATEGORY

In this section we recall some definitions and remarks for a braided monoidal category [4], [10]. A category  $X$  equipped with an object  $I$  and a bifunctor  $\otimes : X \times X \rightarrow X$ , which is associative and for which  $I$  is a twosided identity, we call a *monoidal category*. In this paper objects are linear spaces, an identity  $I$  is a field  $k$  and bifunctors  $\otimes$  means  $\otimes_k$ . We consider only a strict monoidal category  $X$  and denote a set of morphisms for this category by  $\text{Mor}(X)$ . A natural transformation  $\mathcal{B} \in \text{Nat}(\otimes, \otimes^{\text{opp}})$  is a family of morphisms

$$(1) \quad \mathcal{B}_{X,Y} \in \text{lin}(X \otimes Y, Y \otimes X),$$

which for all morphisms  $f \in \text{Hom}(X, Y)$  and  $g \in \text{Hom}(Z, W)$  satisfies the naturality conditions:

$$(2) \quad (g \otimes f) \circ \mathcal{B}_{X,Z} = \mathcal{B}_{Y,W} \circ (f \otimes g).$$

In Particular  $\mathcal{B}_{V,W}$  is  $\mathcal{B}$ -morphism if we have two tetragons

$$(3) \quad \begin{aligned} (\text{id} \otimes \mathcal{B}_{V,W}) \circ \mathcal{B}_{V \otimes W, U} &= (\mathcal{B}_{V,W} \otimes \text{id}) \circ \mathcal{B}_{W \otimes V, U}, \\ (\mathcal{B}_{V,W} \otimes \text{id}) \circ \mathcal{B}_{V \otimes W, U} &= (\text{id} \otimes \mathcal{B}_{V,W}) \circ \mathcal{B}_{W \otimes V, U}, \end{aligned}$$

**Prebraid.** A natural transformation  $\mathcal{B} \in \text{Nat}(\otimes, \otimes^{opp})$  is a *prebraid* if two trigons hold:

$$(4) \quad \begin{aligned} \mathcal{B}_{V \otimes W, U} &= (\mathcal{B}_{V, U} \otimes \text{id}_W) \circ (\text{id}_V \otimes \mathcal{B}_{W, U}), \\ \mathcal{B}_{V, W \otimes U} &= (\text{id}_W \otimes \mathcal{B}_{V, U}) \circ (\mathcal{B}_{V, W} \otimes \text{id}_U), \end{aligned}$$

The consequence of the naturality condition and pairs of trigons is the braided hexagon

**Prebraided monoidal category.** A monoidal category  $X$  is *prebraided* if it is equipped with the prebraid  $\mathcal{B} \in \text{Nat}(\otimes, \otimes^{opp})$ .

A prebraided monoidal category is *braided* if  $\mathcal{B}$  is a natural isomorphism. Then we have pairs of braidings:

$$(5) \quad \mathcal{B} \in \text{Nat}(\otimes, \otimes^{opp}) \quad \text{and} \quad \mathcal{B}^{-1} \in \text{Nat}(\otimes^{opp}, \otimes),$$

such that  $\mathcal{B} \circ \mathcal{B}^{-1} = \text{id}_{\otimes^{opp}}$ ,  $\mathcal{B}^{-1} \circ \mathcal{B} = \text{id}_{\otimes}$ ,  $(B_{U, V})^{-1} = (\mathcal{B}^{-1})_{V, U}$ .

A braided monoidal category is *symmetric* if the braid  $\mathcal{S}$  satisfies the condition:

$$(6) \quad \mathcal{S}_{U, V} \circ \mathcal{S}_{V, U} = \text{id}_{V \otimes U}.$$

This means that the notion  $\mathcal{S} = \mathcal{S}^{-1}$  is badly posed from the categorical point of view.

For the symmetry braid equalled to the flip  $\mathcal{S} = \tau$

$$(7) \quad \tau_{U, V}(u \otimes v) = v \otimes u, \quad u \in U, v \in V.$$

**Lemma 2.1.** Any morphism is a  $\tau$ -morphism in a  $\tau$ -monoidal category.

**Young antisymmetrizer.** Let  $B_n$  be a braid group with generators  $\{\sigma_1, \sigma_2, \dots, \sigma_{n-1}\}$  and relations

$$(8) \quad \begin{aligned} \sigma_i \sigma_j &= \sigma_j \sigma_i & |i - j| > 1, \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} & i = 1, 2, \dots, n-2. \end{aligned}$$

Let  $\mathcal{B}$  be the Yang Baxter operator, i.e. an invertible endomorphism of the  $\Gamma^{\otimes 2}$  which satisfies the braid equation

$$(9) \quad (\text{id} \otimes \mathcal{B})(\mathcal{B} \otimes \text{id})(\text{id} \otimes \mathcal{B}) = (\mathcal{B} \otimes \text{id})(\text{id} \otimes \mathcal{B})(\mathcal{B} \otimes \text{id}) \in \text{End}(\Gamma^{\otimes 3}).$$

Let  $B_k = \text{id}_{k-1} \otimes \mathcal{B} \otimes \text{id}_{n-k-1} \in \text{End}(\Gamma^{\otimes n})$  be the set of endomorphisms of  $\Gamma^{\otimes n}$ . For the Yang Baxter operator  $\mathcal{B}$  we can define the representation  $\rho_{\mathcal{B}}$  of the braid group  $B_n$  in the  $k$ -space  $\Gamma^{\otimes n}$

$$(10) \quad \rho_{\mathcal{B}} \in \text{group}\{B_n, \text{End}(\Gamma^{\otimes n})\} : \quad \rho_{\mathcal{B}}(\sigma_k) = B_k.$$

Let  $\psi$  mean the injection map from a permutation group  $P_n$  to the braid group  $B_n$  [9]. Then we have the image of the map  $\psi$ , the subset  $\Xi_n$

$$(11) \quad \Xi_n = \psi(P_n) \subset B_n.$$

For an Yang Baxter operator  $\mathcal{B}$ , the Young (braided) antisymmetrizer  $Y(\mathcal{B})$  is defined by [1], [5]

$$(12) \quad Y(\mathcal{B}) = \sum_{b \in \Xi_n} \rho_{\mathcal{B}}(b).$$

For  $\mathcal{B} = -\mathcal{S}$  we have the Woronowicz form of the braided antisymmetrizer with a sign of a permutation [9]

$$(13) \quad Y(-\mathcal{S}) = \sum_{b \in \Xi_n} \text{sign}[\psi^{-1}(b)] \rho_{\mathcal{S}}(b).$$

### 3. LIE COALGEBRAS OVER MONOIDAL CATEGORY

Michaelis [6] defined a Lie algebra and a Lie coalgebra over a monoidal category. We recall these definitions and the Koszul theorem in the dual form is rewritten.

**Definition 3.1 (Lie algebra, [6]).** A Lie algebra LA over a monoidal category  $X$  is a pair  $\{\Gamma, m\}$ , where  $\Gamma \in \text{Obj}(X)$  and a map  $m \in \text{Mor}(X)$  is subjected to

- $\tau \circ (m \otimes \text{id}) = (\text{id} \otimes m) \circ \tau_1 \tau_2$ ,
- $J(m) \equiv m \circ (m \otimes \text{id}) \circ (\text{id} + \tau_1 \tau_2 + \tau_2 \tau_1) = 0$ ,
- $\ker Y_2(-\tau) \subset \ker m$ .

Remarks. The first condition is the  $\tau$ -morphism condition for the multiplication. Usually it is omitted, because it is trivial  $\forall m$ , see lemma 2.1. We write this for the pedagogical reasons. The second is the Jacobi identity for the multiplication in the argument free form. For the map  $J(m)$  we have  $J(m) : \Gamma^{\wedge 3} \rightarrow \Gamma$ . The third condition is the commutativity of the multiplication  $m \circ (\text{id} + \tau) = 0$ .

For the Jacobi condition in the free argument form  $J(m) = 0$  we have another possible forms.

**Lemma 3.2.** *The following assertions are equivalent*

- $J(m) \equiv m \circ (m \otimes \text{id}) \circ (\text{id} + \tau_1 \tau_2 + \tau_2 \tau_1) = 0$ ,
- $m \circ (m \otimes \text{id}) - m \circ (\text{id} \otimes m) - m \circ (\text{id} \otimes m) \circ (\text{id} \otimes \tau) = 0$ ,
- $\forall x, y, z \in \Gamma : [[x, y], z] - [x, [y, z]] - [[x, z], y] = 0$ .

**Definition 3.3 (Lie coalgebra, [6]).** A Lie coalgebra LC over a monoidal category  $X$  is a pair  $\{\Gamma, \Delta\}$ , where  $\Gamma \in \text{Obj}(X)$  and  $\Delta \in \text{Mor}(X)$  is subjected to

- $(\Delta \otimes \text{id}) \circ \tau = \tau_2 \tau_1 \circ (\text{id} \otimes \Delta)$ ,
- $cJ(m) \equiv (\text{id} + \tau_1 \tau_2 + \tau_2 \tau_1) \circ (\Delta \otimes \text{id}) \circ \Delta = 0$ ,
- $\text{im } \Delta \subset \text{im } Y_2(-\tau)$ .

Remarks. The first condition is the  $\tau$ -morphism condition for the comultiplication. It is omitted, because it is trivial  $\forall \Delta$  see lemma 2.1. The second is the Jacobi identity for the comultiplication  $\Delta$  in the argument free form. For the co-Jacobiator  $cJ(m)$

we have  $cJ(m) : \Gamma \rightarrow \Gamma^{\wedge 3}$ . The third condition is the cocommutativity of the comultiplication  $(\text{id} + \tau) \circ \Delta = 0$ .

Consider an exterior algebra  $\mathcal{E}_\tau(\Gamma^*) = \{\wedge \Gamma^*, \hat{d}_\Delta\}$  over a Lie algebra  $\text{LA} = \{\Gamma, m\}$ . This means that the equation  $\hat{d}_\Delta|_{\Gamma^*} = m^*$  holds.

**Theorem 3.4 (Koszul Theorem, [3]).** *For an exterior algebra  $\mathcal{E}_\tau(\Gamma^*)$  over Lie algebra  $\text{LA}$  the following assertions are equivalent*

- $J(m) = 0 \in \Gamma$ ,
- $(\hat{d}_\Delta)^2 = 0 \in \Gamma^{\wedge 3}$ .

*Proof.* Let  $\alpha \in \Gamma^*$  and  $x, y, z \in \Gamma$ . Let us introduce two operators  $e_x y = x \wedge y$ ,  $(i_x \alpha)y = \alpha(e_x y)$ . Then the Lie derivation is  $\mathcal{L}_X = \hat{d} \circ i_X + i_X \circ \hat{d}$ . For  $(\hat{d}_\Delta)^2 = 0$  we have

$$\begin{aligned} 0 &= \langle (\hat{d})^2 \alpha, x \wedge y \wedge z \rangle = \langle i_X \hat{d}^2 \alpha, y \wedge z \rangle \\ &= \langle (i_X \circ \hat{d}) \hat{d} \alpha, y \wedge z \rangle = \langle (\mathcal{L}_X - \hat{d} \circ i_X) \hat{d} \alpha, y \wedge z \rangle \\ &= - \langle \hat{d} \circ i_X \hat{d} \alpha, y \wedge z \rangle + \langle \alpha, \mathcal{L}_X(y \wedge z) \rangle. \end{aligned}$$

Consider two equations

$$\begin{aligned} \langle \hat{d} \circ i_X \hat{d} \alpha, y \wedge z \rangle &= \langle i_X \hat{d} \alpha, m(y \wedge z) \rangle = \langle \hat{d} \alpha, e_X m(y \wedge z) \rangle \\ &= \langle \alpha, m(x \wedge m(y \wedge z)) \rangle = \langle \alpha, [x, [y, z]] \rangle, \\ \langle \alpha, \mathcal{L}_X(y \wedge z) \rangle &= \langle \alpha, (\mathcal{L}_X(y)) \wedge z + y \wedge (\mathcal{L}_X(z)) \rangle \\ &= \langle \hat{d} \alpha, [x, y] \wedge z + y \wedge [x, z] \rangle = \langle \alpha, [[x, y], z] + [y, [x, z]] \rangle. \end{aligned}$$

Then

$$\langle \hat{d}^2 \alpha, x \wedge y \wedge z \rangle = \langle \alpha, [[x, y], z] + [y, [x, z]] - [x, [y, z]] \rangle = 0. \blacksquare$$

For a derivation of the free tensor algebra  $\text{TF}$  we have a factor derivation  $\hat{d} \in \text{TF} / \ker Y(-\tau)$ .

**Lemma 3.5 (Factor derivation).** *The factor derivation  $\hat{d}_\Delta \in \text{der}(\text{TF} / \ker Y(-\tau))$  exists for any derivation  $d_\Delta$  of a free algebra  $\text{TF}$ .*

*Proof.* This follows from the fact that any comultiplication  $\Delta$  is the morphism in the  $\tau$ -monoidal category. Using the recurrent formula for the derivation  $d_\Delta$

$$d_\Delta | \Gamma^{\otimes n} = \sum_{k=1}^{n-1} (-1^k) \text{id}_{k-1} \otimes \Delta \otimes \text{id}_{n-k-1}.$$

we can proof this fact by induction.  $\blacksquare$

We can consider an exterior algebra  $\mathcal{E}_\tau(\Gamma) = (\wedge \Gamma, \hat{d}_\Delta)$  over a Lie coalgebra  $\text{LC} = (\Gamma, \Delta)$ . Then the equation  $\hat{d}_\Delta|_\Gamma = \Delta$  holds.

**Theorem 3.6 (Koszul Theorem for Lie coalgebras).** *For an exterior algebra  $\mathcal{E}_\tau(\Gamma)$  over a Lie coalgebra LC the following assertions are equivalent:*

- $\hat{d}_\Delta^2 = 0 \in \Gamma^{\wedge 3}$ ,
- $\text{im}(d_\Delta)^2 \subset \ker Y(-\tau)$ ,
- $cJ(\Delta) = (\text{id} + \tau_1\tau_2 + \tau_2\tau_1) \circ (\Delta \otimes \text{id}) \circ \Delta = 0$ ,
- $(\Delta \otimes \text{id}) \circ \Delta - (\text{id} \otimes \Delta) \circ \Delta - (\text{id} \otimes \tau) \circ (\Delta \otimes \text{id}) \circ \Delta = 0$ .

*Proof.* The first and the second conditions are equivalent from the lemma 3.5

$$Y_3(-\tau) \circ d_\Delta^2 | \Gamma = 0 \Leftrightarrow \hat{d}_\Delta^2 = 0.$$

The remark is that for a derivation we can check conditions on generators of the algebra. From the identity in lemma 4.5 the second condition is equivalent to the fourth condition. From this condition by the commutativity of the multiplication  $m \circ \tau = -m$  we get the third condition. ■

Remark. The third condition is the Jacobi formula for the comultiplication  $\Delta$  in the Woronowicz form [9].

#### 4. S-LIE COALGEBRAS

Let us recall the definition of the S-Lie algebras introduced by D. Gurevich [2]. In this section  $\mathcal{S} \in \text{End}(\Gamma^{\otimes 2})$  is a symmetry braid,  $\mathcal{S}^2 = \text{id}$ . A category  $X$  is  $\mathcal{S}$ -braided.

**Definition 4.1 ( $\mathcal{S}$ -Lie algebra, [2]).** An S-Lie algebra SLA over  $\mathcal{S}$ -braided category  $X$  is a pair  $(\Gamma, m)$ , where  $\Gamma \in \text{Obj}(X)$  and the multiplication  $m \in \text{Mor}(X)$  is subjected to

- $\mathcal{S} \circ (\text{id} \otimes m) = (m \otimes \text{id}) \circ \mathcal{S}_1 \mathcal{S}_2$
- $m \circ (m \otimes \text{id}) \circ (\text{id} + \mathcal{S}_1 \mathcal{S}_2 + \mathcal{S}_2 \mathcal{S}_1) = 0$ ,
- $\ker m \subset \ker Y_2(-\mathcal{S})$

Remarks. For the first condition if  $\mathcal{S}$  is the symmetry braid then the following morphism conditions for the multiplication are equivalent

$$\mathcal{S} \circ (m \otimes \text{id}) = (\text{id} \otimes m) \circ \mathcal{S}_2 \mathcal{S}_1 \Leftrightarrow \mathcal{S} \circ (\text{id} \otimes m) = (m \otimes \text{id}) \circ \mathcal{S}_1 \mathcal{S}_2.$$

The first condition is the strong condition for multiplications. For example

$$(14) \quad \mathcal{S}(e_i \otimes e_j) = \epsilon_{ij} \cdot e_j \otimes e_i, \quad \epsilon_{ij}\epsilon_{ji} = 1, \quad \mathcal{S}^2 = \text{id}.$$

Then the comultiplication  $\Delta$  is  $\mathcal{S}$ -morphism for color algebras [8].

**Definition 4.2 ( $\mathcal{S}$ -Lie coalgebra).** An S-Lie coalgebra SLC over  $\mathcal{S}$ -braided monoidal category  $X$  is a pair  $(\Gamma, \Delta)$ , where  $\Gamma \in \text{Obj} X$  and the comultiplication  $\Delta \in \text{Mor}(X)$  is subjected to

- $(\Delta \otimes \text{id}) \circ \mathcal{S} = \mathcal{S}_1 \mathcal{S}_2 \circ (\text{id} \otimes \Delta)$ .
- $cJ(\Delta) = (\text{id} + \mathcal{S}_2 \mathcal{S}_1 + \mathcal{S}_1 \mathcal{S}_2) \circ (\Delta \otimes \text{id}) \circ \Delta = 0$ ,
- $\text{im } \Delta \subset \ker(Y_2(-\mathcal{S}))$ .

Now we consider the necessary and sufficient condition for the existence a factor derivation of a factor algebra  $T\Gamma / \ker Y(-\mathcal{S})$ .

**Lemma 4.3 (Identities for  $Y(-\mathcal{S})$ ).** *For the Young antisymmetrizer  $Y_3(-\mathcal{S})$  we have the following identity*

$$Y_3(-\mathcal{S}) \circ \mathcal{S}_1 \circ \mathcal{S}_2 = Y_3(-\mathcal{S}) \circ \mathcal{S}_2 \circ \mathcal{S}_1 = Y_3(-\mathcal{S}).$$

For the Young antisymmetrizer  $Y(-\mathcal{S})$  a derivation of free tensor algebra  $d \in \text{der}(T\Gamma)$  should satisfy condition imposing that it will be a factor derivation  $\hat{d} \in \text{der}(T\Gamma / \ker Y(-\mathcal{S}))$ .

**Lemma 4.4 (Factor derivation).** *Let a comultiplication  $\Delta$  be  $\mathcal{S}$ -morphism and for a derivation of a free algebra let  $d_\Delta | \Gamma = \Delta$ . Then exist a factor derivation  $\hat{d}_\Delta \in \text{der}(T\Gamma / \ker Y(-\mathcal{S}))$ ,*

$$(\Delta \otimes \text{id})\mathcal{S} = \mathcal{S}_{1,2}(\text{id} \otimes \Delta) \quad \Rightarrow \quad d_\Delta \ker Y(-\mathcal{S}) \subset \ker Y(-\mathcal{S}).$$

*Proof.* We can write the condition  $d_\Delta \ker Y(-\mathcal{S}) \subset \ker Y(-\mathcal{S})$  in the following form for a derivation  $d | \Gamma^{\otimes 2}$

$$Y_3(-\mathcal{S}) \circ (\Delta \otimes \text{id} - \text{id} \otimes \Delta) \circ (\text{id} + \mathcal{S}) = 0.$$

We can proof this equation using the morphism condition and the lemma 4.3. Using these results we get

$$[Y_3(-\mathcal{S}) - Y_3(-\mathcal{S}) \circ \mathcal{S}_1 \mathcal{S}_2] \circ (\Delta \otimes \text{id}) - [Y_3(-\mathcal{S}) \circ \mathcal{S}_2 \mathcal{S}_1 - Y_3(-\mathcal{S})] \circ (\text{id} \otimes \Delta) = 0$$

■

The inverse is not true. For the model (14) of the symmetry  $\mathcal{S}$  the condition  $d_\Delta \ker Y(-\mathcal{S}) \subset \ker Y(-\mathcal{S})$  are not equivalent to the morphism condition for  $\Delta$ .

**Lemma 4.5 (Identity for  $\Delta$  and  $\mathcal{S}$ ).** *Let the multiplication  $\Delta$  be  $\mathcal{S}$ -morphism. Let  $\Delta_{\mathcal{S}} = \Delta - \mathcal{S} \circ \Delta$ . Then we have the following identity*

$$\begin{aligned} Y(-\mathcal{S}) \circ (\Delta \otimes \text{id} - \text{id} \otimes \Delta) \\ = [(\Delta_{\mathcal{S}} \otimes \text{id}) \circ \Delta_{\mathcal{S}} - (\text{id} \otimes \Delta_{\mathcal{S}}) \circ \Delta_{\mathcal{S}} - (\text{id} \otimes \mathcal{S}) \circ (\Delta_{\mathcal{S}} \otimes \text{id}) \circ \Delta_{\mathcal{S}}] \circ (\text{id} - \mathcal{S}). \end{aligned}$$

*Proof.* Using the  $\mathcal{S}$ -morphism condition for the comultiplication and the ansatz with  $\Delta_{\mathcal{S}}$  above condition will be obtained by simple calculations. ■

**Theorem 4.6 (Koszul Theorem for  $\mathcal{S}$ -Lie coalgebras).** *For an  $\mathcal{S}$ -exterior algebra  $\mathcal{E}_{\mathcal{S}}(\Gamma)$  over an  $\mathcal{S}$ -Lie coalgebra SLA the following assertions are equivalent*

- $\hat{d}^2 = 0 \in \Gamma^{\wedge 3}_{\mathcal{S}}$ ,
- $\text{im}(d_\Delta)^2 \subset \ker Y_3(-\mathcal{S})$ ,
- $(\text{id} + \mathcal{S}_{1,2} + \mathcal{S}_{2,1}) \circ (\text{id} \otimes \Delta) \circ \Delta = 0$ ,
- $((\Delta \otimes \text{id})\Delta - (\text{id} \otimes \Delta) \circ \Delta - (\text{id} \otimes \mathcal{S}) \circ (\Delta \otimes \text{id}) \circ \Delta = 0$ .

*Proof.* If a comultiplication  $\Delta$  is  $\mathcal{S}$ -morphism then the first and the second condition are equivalent from the lemma 4.4

$$Y_3(-\mathcal{S}) \circ d_\Delta^2 | \Gamma = 0 \Leftrightarrow \hat{d}_\Delta^2 = 0.$$

From the identity in lemma 4.5 the second condition is equivalent to the fourth condition, the Jacobi condition in the Woronowicz form. From this condition by the  $\mathcal{S}$ -commutativity of the comultiplication  $\Delta$  we can get the third condition. ■

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#### REFERENCES

1. M. Bozejko and R. Speicher, Completely positive maps on Coxeter groups, deformed commutation relations, and operator spaces, *Math. Ann.* 300 (1994) 97–120.
2. D. Gurevich, The Yang-Baxter equations and a generalization of formal Lie theory, *Soviet Math. Doklady* 33 (1986) 758–762.
3. J. Koszul, Homologie et cohomologie des algèbres de Lie, *Bull. Soc. Math. France* 78 (1950), 1–63.
4. S. MacLane, *Categories for the Working Mathematician*, Springer Verlag, 1971.
5. N. Metropolis and G.-C. Rota, Symmetry Classes: Function of Three Variables, *American Mathematical Monthly* 98 No. 4 (1991) 328–332.
6. W. Michaelis, Lie coalgebras, *Advances in Mathematics* 38 (1980), 1–54.
7. Z. Oziewicz, E. Paal and J. Różański, Coalgebras, cocompositions and cohomology, in ‘Non Associative Algebra and Its Applications’, in series *Mathematics and its Applications*, ed. S. Gonzalez, Kluwer Acad. Publishers, str. 314–322, 1994.
8. Scheunert M., Generalized Lie algebras, *J. Math. Phys.* 20 (1979).
9. S. L. Woronowicz, Differential calculus on compact matrix pseudogroups (quantum groups), *Commun. Math. Phys.* 122 (1989), 125–170.
10. D. Yetter, Quantum groups and representations of monoidal categories, *Math. Proc. Camb. Phil. Soc.* 108 (1990).

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